Answer all the questions. Each question is worth 5 points. You may state correctly and use any result proved in the class. However if an answer is an almost immediate consequence of the stated result, such a result also need to be proved.

All topological spaces are assumed to be Hausdorff.

1) Let X be a complex normed linear space. Let $f : X \to C$ be a non-zero linear map . Show that either $f(\{x \in X : \|x\| \leq 1\})$ is a bounded set or is all of C. In the second case show that $ker(f)$ is dense in X.

2) Show that for any Banach space X,

$$\bigoplus_{\infty} X = \{\{x_n\}_{n\geq 1} : x_n \in X,\ sup_{n\geq 1}\|x_n\| < \infty\}$$

equipped with the norm $\|\{x_n\}_{n\geq 1}\| = sup_{n\geq 1}\|x_n\|$ is a Banach space.

3) Let $M = \{f \in C([0,1]) : f([0, \frac{1}{2}]) = 0\}$. On the quotient space, let $\Phi : C([0,1])|M \to C([0, \frac{1}{2}])$ be defined by $\phi([f]) = f|[0, \frac{1}{2}]$. Show that Φ is a well-defined, liner, onto, isometry.

4) Let X, Y be a LCTVS spaces. Let $T : X \to Y$ be an isomorphism. Suppose X^* and Y^* are equipped with the weak*-topology. Show that $T^* : Y^* \to X^*$ is an isomorphism.

5) Give examples of two normed linear spaces, and a continuous linear map T between them such that T^* has closed range but the range of T is not closed.

6) Construct a sequence $f_n : [0,1] \to [0,1]$ of continuous functions such that $\|f_n\| = 1$ for all $n \geq 1$, $f_n(t) \to 0$ for all $t \in [0,1]$. Use the Riesz representation theorem to show that $f_n \to 0$ in the weak topology of $C([0,1])$.

7) Let $D = \{z : |z| < 1\}$ be the open unit Disc. Let $A(D)$ denote the space of analytic functions on D with the family of semi-norms, $p_z(a) = |a(z)|$ for $z \in D$ and $a \in A(D)$. Let $F = \{p \in A(D) : p\ is\ a\ polynomial\ of\ degree\ atmost`n\}$. Show that $A(D) = F \bigoplus Y$ (direct sum) for some closed subspace $Y \subset A(D)$.

8) Let X be a LCTVS space. Let $C \subset X$ be a closed convex set. Show that C is also closed in the weak-topology.